

Classroom Factors Related to Geometric Proof Construction Ability¹

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Abstract: *Although proof and reasoning are seen as fundamental components of learning mathematics, research shows that many students continue to struggle with geometric proofs. In addition, there is limited research on the link between the teaching and learning of proof. Our research focuses on aspects of the classroom microculture, teacher's pedagogical choices, and how these may impact student proof-construction ability in geometry. We present quantitative and interview data that highlight students' strengths and weakness with various aspects of formal proof. Analyses of classroom episodes illustrate some of the social norms, sociomathematical norms, and classroom practices that may influence student proof-construction ability. Although, teacher's choices helped students develop an appreciation of the importance of diagrams and attention to detail, aspects of the pedagogy leave students ill-prepared to formulate logical arguments on their own.*

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Introduction

Proof is fundamental to the discipline of mathematics because it is the convention that mathematicians use to establish the validity of mathematical statements. In addition, the teaching of proof as a sense-making activity is fundamental to developing student understanding in geometry and other areas of mathematics. Despite the fact that student difficulty with proof has been well established in the literature, existing empirical research on pedagogical methods associated with the teaching and learning of geometric proof is insufficient (Chazan, 1993; Hart, 1994; Herbst, 2002; Martin & Harel, 1989). Our work in this area has begun to address the need for research into the pedagogy of geometric proof instruction. We focus on formal geometric proof because high school geometry is traditionally the course in which students are first required to construct proofs. As well, formal proofs are typically what students in many parts of the world see and do as part of a proof-based geometry course.

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In this paper we summarize findings that connect student ability to construct proof in geometry and classroom factors that may influence that ability. More specifically, we address two objectives:

1. To characterize the psychological aspects of students' evolving proof-construction ability in proof-based geometry classes in order to update and expand existing research in this area;
2. To link students' geometric proof-construction ability to the classroom microculture as well as to teachers' pedagogical choices.

This work is part of a larger three-year study that focuses on several aspects of student understanding of proof in geometry and pedagogical influences on student learning in this domain.

Theoretical Framework

The theoretical framework for our research takes into account student understanding of proof as well as the social factors that may influence understanding. For this study we focus on two important factors, namely, the classroom microculture and the teacher's pedagogical choices (Simon, 1995; Stein, Grover, & Henningsen, 1996; Yackel & Cobb, 1996). Figure 1 illustrates the components of the microculture and pedagogical choices, and how these may influence student proof construction ability.

The theoretical lens through which we view students' mathematical development and the factors influencing that development is associated with the emergent perspective as described by Cobb and Yackel (1996). This framework is useful in that it attempts to describe individual learning in the social context of the classroom. Thus, student ability to construct proofs is seen as constructed on both a psychological level and a social level. From a psychological perspective, several researchers have characterized students' understanding of proof in terms of their reasoning ability and formal proof-construction ability (Balacheff, 1991; Harel & Sowder, 1998; Senk, 1985). In this paper, we focus on students' ability to perform increasingly difficult formal proof-construction tasks. We assessed the level of difficulty of formal proof tasks using a modification of the difficulty levels described by Senk.

From a social perspective, elements of the classroom microculture as described by Cobb (1999) also influence students' ability to construct proofs. In particular, social norms, sociomathematical norms, and classroom mathematical practices comprise the classroom microculture in which the expectations and collective understanding about what it means to write proofs is continually being constituted. Social norms are defined as normative aspects of the classroom that may apply to any subject

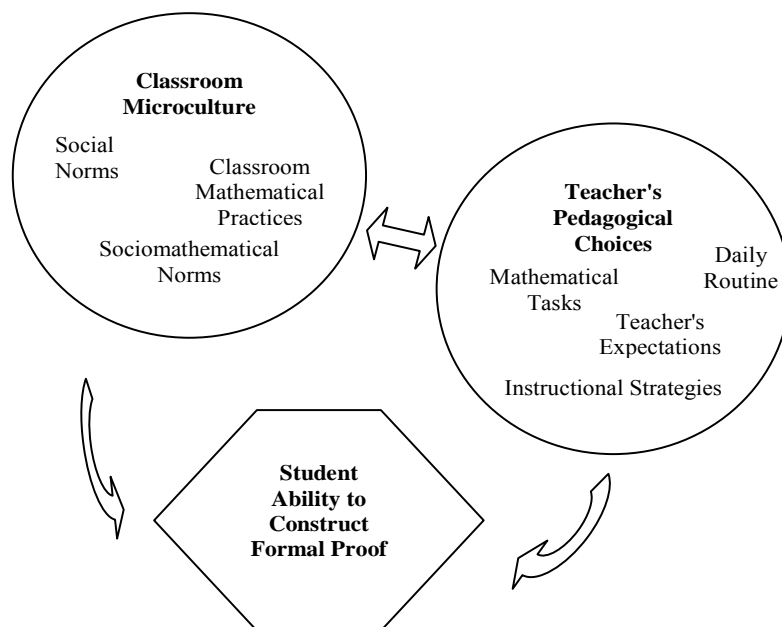


Figure 1. Social factors influencing student understanding of proof.

area, such as the expectation that all solutions must be justified. Sociomathematical norms refer to evolving classroom norms that specifically refer to mathematics, such as what counts as a unique mathematical solution or what is thought to be a clever problem solving strategy. Classroom mathematical practices refer to taken-as-shared practices that relate to specific mathematical ideas such as acceptable methods for proving triangles congruent (Cobb, 1999; Cobb & Yackel, 1996).

We also focus on teachers' pedagogical choices, recognizing the significant impact these choices may have on the evolving microculture of the classroom as well as on students' opportunity to learn how to construct proofs. We define pedagogical choices to include the choice of mathematical tasks, the daily routine, the instructional strategies, and the teachers' expectations about student ability.

Methods

During the first year of the study, we collected data in two proof-based geometry classes taught by two different teachers in a large high school in the mid-western United States. Researchers observed and videotaped the two geometry classes

almost daily for the four months in which proof was a major focus of the curriculum.

To assess the students' ability to construct proofs, we administered a performance assessment instrument, the Proof Construction Assessment, to all students in the two participating classes. The Proof Construction Assessment included items at varying levels of difficulty (Senk, 1985). Items at the first level required students to fill in the blanks in a partially constructed two-column proof. Items at the second level of difficulty addressed students' understanding of conditional statements. The local deductions items, level three, assessed students' ability to draw one valid conclusion from a given statement and to justify the conclusion. Items at the fourth level of difficulty required students to engage in multi-step reasoning with hints provided. Items at the fifth level required students to independently generate a complete, multi-step formal proof. In addition to some original items, the instrument includes items modified from Senk (1985) and from the Third International Mathematics and Science Study (TIMSS) (1995).

Clinical interviews were conducted with ten focus students in order to clarify their written responses on the Proof Construction Assessment and to further assess their proof-writing abilities. In particular, students were asked to compare and assess several student-written proofs and to create at least one original proof during the interview.

The Proof Construction Assessment and the clinical interviews provided two lenses with which to interpret students' individual proof-construction ability. In addition, observations, video recordings and other data helped us to learn about the social context for the development of proof-construction ability in order to interpret this information and connect the social aspects of students' experiences to the psychological abilities they displayed.

Results and Discussion

In this section, we describe the results from the Proof Construction Assessment and clinical interviews in order to characterize the psychological aspects of students' evolving proof construction ability. We also describe aspects of the classroom microculture and the participating teachers' pedagogical choices and discuss how these factors may have influenced students' ability to write formal proofs.

Proof Construction Ability

Results from the Proof Construction Assessment are found in Table 1. The table is organized by difficulty level in order to demonstrate the relationship between level of difficulty and student performance. Student performance was poorest on items 2 and 4, which were assessed as the most difficult. These items required students to

write original proofs of statements based on given conditions. Even though students also needed to write a proof for item 5 (see Figure 2), they were provided with ideas for outlining the proof. Strong student scores on item 5 also may be due to students' familiarity with the content (similar triangles), as the performance assessment instrument was administered shortly after the completion of a unit on similar triangles.

Table 1
Student Performance on the Proof Construction Assessment.

Item Number	Item Type (level of difficulty)	Avg. Score as a percent for Mrs. Anderson's students	Avg. Score as a percent for Mrs. Betts' students
1	Fill in proof (1)	61.6	69.5
3	Conditional statements (2)	66.5	76.7
6	Local deductions (3)	42.3	39.0
5	Proof with hints (4)	52.3	70.5
4	Unsupported synthetic proof (5)	33.1	30.5
2	Unsupported analytic proof (5)	22.3	27.4
Total		46.7	51.1

Note: () indicates level of difficulty given for the item, where 1 is easiest and 5 is most difficult.

Although item 6 (see Figure 3), local deductions, was classified as level 3, student performance was poorer on this item than on item 5 (Figure 2), proof with hints, which was classified as level 4. In comparing items 5 and 6, it should be noted that item 6 is completely open-ended whereas item 5 contains an outline of the proof. More specifically, to successfully complete item 6, students must draw a single deduction from given information. Even though multiple deductions are required for item 5, students know what the string of deductions should lead to, and they are also provided with geometric properties that may be key components of the argument. Thus, the open-ended nature of item 6 appears to be a better factor for determining level of difficulty for the students who participated in the study.

Student performance was best on items 1 and 3, which were assessed to be at the two lowest difficulty levels. For item 1, students were asked to fill in the missing statements or reasons for a proof that had been developed for them. For item 3, students were required to write a conditional statement and then use this statement to determine what information was given and what was necessary to prove if asked to justify the conditional statement.

During clinical interviews, many students claimed that an inability to recall definitions, theorems, and postulates made it difficult for them to assess or produce

Given: Quadrilateral \overline{KLMN} is a parallelogram.
 Segments \overline{LQ} and \overline{KN} intersect at P.
 N is on line \overleftrightarrow{MQ}

Prove: $\triangle KLP$ is similar to $\triangle NQP$

Several hints about how this proof may be constructed are provided below. Please use some of these hints to write a valid proof that $\triangle KLP$ is similar to $\triangle NQP$.

Hints:
 Recall that one way of proving that triangles are similar requires showing that two pairs of corresponding angles in the triangles are congruent. Use the quadrilateral to identify a pair of parallel lines. Use properties of parallel lines and related angles to identify pairs of congruent angles. You may also find other congruent angles in the diagram.

Figure 2. Proof Construction Assessment Item 5.

For each part, write a logical conclusion that follows from the given set of conditions. Also, record a reason that supports each conclusion.

a. Given: Three distinct points A, B, and C lie on a line. $AB=BC$.
 Conclusion: _____
 Reason: _____

b. Given: \overline{XY} intersects \overline{ZW} at point P. Point P is between X and Y. Point P is between Z and W.
 Conclusion: _____
 Reason: _____

Figure 3. Proof Construction Assessment Item 6 – parts a and b.

proofs both on the Proof Construction Assessment and in the interview. When the geometric content was somewhat familiar to the students, they were able to talk through aspects of the given information (or provide a diagram) and eventually provide at least a start of an outline for a proof. When the content was unfamiliar, most students had difficulty building chains of reasoning or even providing one valid conclusion from a given statement.

A second important result from the interviews was the students' stated preference for two-column proofs rather than other formats that had been shown in class, such as flow-chart or paragraph proofs. On the Proof Construction Assessment, students almost exclusively used the two-column format to write proofs. When students were asked to rank similar proofs in paragraph form and two-column form, all students ranked the two-column proof higher, although the logical arguments presented in both formats were essentially equivalent.

Pedagogical choices

Both teachers followed the order and scope of the textbook quite closely. The typical daily routine involved discussing homework assigned from the textbook, demonstrating new material, and practicing new material on textbook-like worksheets. The introduction of new ideas typically was teacher-directed through lecture and questioning.

Although the two classrooms were similar in many respects, the difference in number of years of experience, and, perhaps, resulting familiarity with content, allowed the more experienced teacher to respond more flexibly to classroom situations and to students. For example, Mrs. Betts, who had more than 20 years of experience as a teacher, encouraged student-to-student discussions of homework and in-class assignments. (This name and all others that follow are pseudonyms.) Although Mrs. Betts' class discussions were generally teacher-led, she was more willing to follow up on students' mathematical suggestions that strayed from the planned solution. In contrast, Mrs. Anderson, who had been teaching mathematics for five years, gave students very little opportunity to discuss proofs with each other. Mrs. Anderson rarely strayed from teacher-directed activities and exhibited errors in her reasoning and in structuring logical arguments more often than Mrs. Betts.

Through interviews and informal conversations with researchers, both teachers shared their belief that students needed to have proofs demonstrated, because constructing proofs independently was too difficult for the students. As a result, teachers expected students to emulate the teacher's approach for solving problems and writing proofs.

Episode 1 illustrates how some of Mrs. Anderson's expectations were manifested in her instructional strategies and in her selection of mathematical tasks. After presenting one or two new definitions, theorems, or postulates, both teachers typically assigned in-class worksheets and textbook homework that included several proofs applying the new facts. In Episode 1, Mrs. Anderson handed out a worksheet containing several proofs that were similar in format, length, and logical reasoning pattern. Before allowing students to get started on the worksheet, Mrs. Anderson demonstrated one of the proofs on the worksheet for them.

Episode 1

Mrs. A: There's only seven of them and then the bonus. I think I'm gonna try just doing one with you and see what you can do with these on your own...Remembering that in order to prove the lines parallel you have to focus on one of those special pairs of angles. If you can find something out about the alternate interior angles or the corresponding angles or the consecutive interior angles, those are the ones that you know something about. That you know the relationships exist if you have parallel lines. So you'd be enforcing the converse of those theorems by trying to prove the lines parallel in that way. Let's look at the first one.

By demonstrating one of the proofs on the worksheet and describing the logical reasoning pattern required to construct all the proofs on the worksheet, Mrs. Anderson restricted the scope of the task. Students were not required to construct a reasoned argument. Rather, they were left only to fill in the details of already structured arguments. This pattern of modeling proof "types" before asking students to complete similar proofs was typical for both teachers.

Classroom Microculture

Because many social norms, sociomathematical norms, and classroom mathematical practices are established through choices made by the teacher, our discussion of the classroom microculture reflects the reciprocal influences of classroom microculture and pedagogical choices. For example, one prevalent social norm that was potentially related to students' proof construction ability was the sense that the teacher was the authority. In both classrooms, teachers were the focal point of instruction. The teachers took primary responsibility for constructing convincing arguments and validating student responses. During class discussions, students' responses were typically one word or phrase, which the teacher expanded upon in order to create a full justification (see Episode 3 for another example).

Another social norm was the expectation that all solutions be justified. In both classrooms, it was not sufficient to make a statement without an accompanying reason. When students described geometric relationships in a problem or proof,

teachers typically asked, “Why?” “How do you know?” or “What’s the reason?” Although it was often the teacher who provided the reason, students were aware that a reason was required.

A sociomathematical norm that was constituted in the classroom was the sense that a valid solution was one that was presented using mathematical conventions, whether or not the reasoning was correct. As the semester progressed, students presented their solutions using the convention preferred by the teacher (e.g., two-column proofs with certain acceptable abbreviations), and subsequently indicated a preference for solutions in two-column format when asked to critique others’ work.

There were three classroom mathematical practices that became taken-as-shared understandings about proof in geometry. The first of these mathematical practices is the importance of details in proof writing. The second is the understanding that only certain methods are valid for establishing the congruence of overlapping triangles. Third, students came to accept that marking diagrams is an essential part of the proof-writing process. Short episodes from the classrooms are included to give the reader a sense of typical classroom conversations and how the mathematical practices were manifested by the teacher and students.

In Episode 2, Mrs. Betts reviewed a proof that a student had written on the board. In one part of the proof, the student, Hallie, was given that two pairs of segments (\overline{DE} and \overline{EG} , \overline{BF} and \overline{EG}) were perpendicular (see Figure 4). Hallie’s conclusion was that $\angle DEF$ and $\angle BFG$ were congruent. Although Mrs. Betts did not comment on the general reasoning or the validity of the proof, she noted that there seemed to be some missing steps related to the congruent angles. She invited all students to help Hallie figure out what should be written in order to fill in the missing details of the proof.

Episode 2

Nathan: Can we say... you know that $\angle DEF$ is congruent to $\angle BFG$?

Mrs. B: The question is ‘how do you know they are congruent?’ What information do you have that’s letting you say they are congruent? All the information you have has been listed here, so far.

Hallie: I should have said something like, because these two segments were perpendicular, then the angles are congruent.

Mrs. B: OK. Let’s go back to that statement. If these two segments are perpendicular, then... how do you finish that?

Hallie: Then... (unsure how to respond)

Nathan: They form right angles.

Mrs. B: Yeah, but that's not said here yet [in the proof]. You haven't said that they form right angles yet. So before you say the angles are congruent, you have some things you have to insert here.

After making this last statement, Hallie returned to the board to fill in the details as requested by Mrs. Betts. Mrs. Betts made no further comments on the proof as a whole or on Hallie's added details. Both Mrs. Betts and Mrs. Anderson spent considerable time with the students to help them get the details of a proof. This appeared to be important to the teachers in the proof-writing process, and became an important part of what the students focused on in their proofs and proofs written by others.

In Episode 3 we encounter Mrs. Anderson near the end of class time. She was introducing the homework assignment and what was required of the students. With very little student input, she verbally walked through the first of the assigned problems (see Figure 5).

Episode 3

Mrs. A: What I've given you are several more proofs... What you need to do is tackle these the same way you've tackled the proofs you've done for the last three days... In the first one, mark angle P and angle S as the congruent angles. That's given. And then mark segment PQ and SQ as congruent segments. What else in the figure is important? What else is there that you think you should identify that's going to help you make those two triangles congruent?

Dottie: Q

Mrs. A: Yeah, angle Q. That reflexive angle again is showing up shared in both triangles. So you would identify that as being angle Q. Angle PQS congruent to angle...Let's see. PQR is congruent to angle SQT, by the reflexive property. You could show the triangles congruent by ASA. Some of them [assigned proofs] stop with congruent triangles. Some of them ask you to go on and show some corresponding parts. None of these proofs are extremely lengthy. They're probably 4 or 5 step proofs.

As in Episode 1, Mrs. Anderson provided an outline for this proof and suggested that students follow it for the other exercises on the worksheet. In this case, Mrs. Anderson modeled the method for proving overlapping triangles congruent and

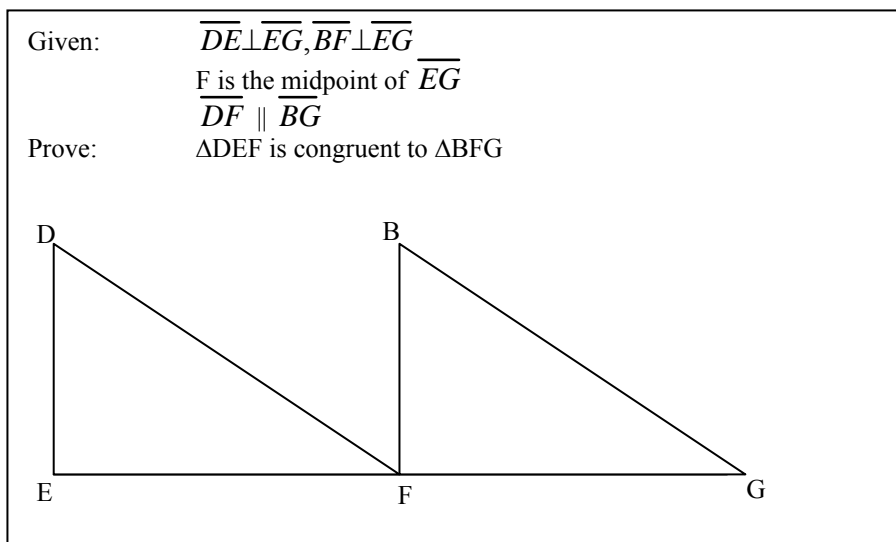


Figure 4. Homework problem associated with Hallie's proof.

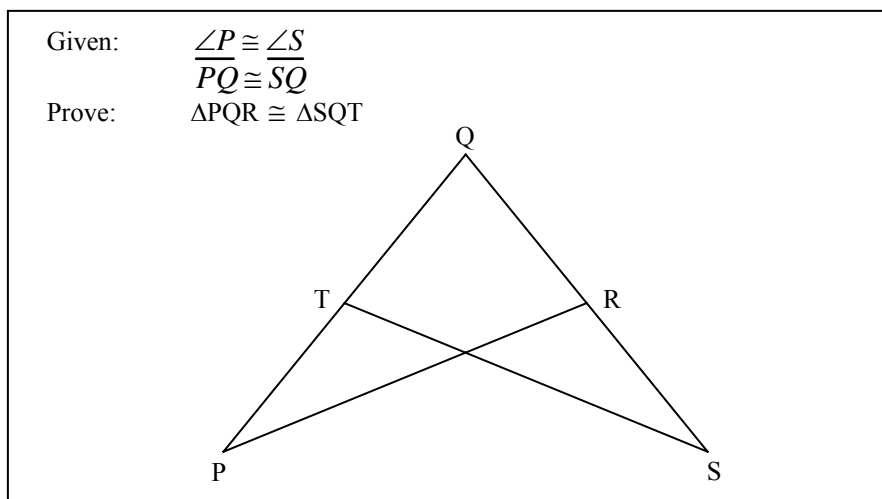


Figure 5. Congruent overlapping triangles problem.

drew upon an already taken-as-shared method for proving any two triangles congruent. A second classroom mathematical practice illustrated in Episode 3 is the expectation that students mark their diagrams in prescribed ways in order to record information that will be used in the proof. In this episode, Mrs. Anderson tells the students exactly what should be marked. During interviews with the focus students, many commented on diagram marking as an important part of the proof writing process.

In both Episodes 2 and 3, we see aspects of the microculture as well as teacher's pedagogical choices. In Episode 2, the teacher emphasized the importance of justifying every step in a proof as well as the critical need to focus on the details. The teacher did not commend the students for laying out a reasonable logical argument, but rather, emphasized the need to have each step follow from the preceding step. In Episode 3, the teacher-led discussion indicated to students that they did not need to take much responsibility for structuring a proof. In class, it was almost always the teacher who had the mathematical authority to produce and assess the validity of proofs. Students rarely had to invent the pattern of reasoning for a proof. Much of the reasoning was done for the students by the textbook, that grouped isomorphic proofs together, and the teacher who provided the sample on which to pattern the practice proof exercises. The students' job was to focus on the details. Perhaps, these aspects of the microculture, along with the teachers' pedagogical choices, account for students' ability to complete fill-in proofs and construct a proof with hints provided, and their inability to draw conclusions independently or construct proofs without hints.

Conclusions

Our finding that students have great difficulty generating complete proofs echoes the findings of Senk (1985) and others (as reported in Hart, 1994; Healy & Hoyles, 1998). It is our investigation of the classroom microculture and teachers' pedagogical choices and their connection to students' proof construction ability that sets our work apart from the existing literature. Here we summarize several factors discussed in this paper that may have influenced student performance on proof construction items.

The examination of teachers' pedagogical choices illuminates various factors that contributed to taken-as-shared classroom practices related to constructing proofs. Three of these choices are: (a) the teachers' choice to use materials that group proofs that require a particular strategy, (b) the teachers' choice to demonstrate or provide a model of the needed proof strategies before setting students to work, and (c) the teachers' choice to focus on the details of a proof more than the overall logical structure of the proof.

As a result of the participating teachers' choices, several related norms and mathematical practices developed in the research classrooms. For example, students were aware of how the proof-writing process was supposed to look: first mark the diagram, and then write a formal proof in two-column format. However, students were not responsible for producing proofs other than those that fit a format or strategy described by the teacher. Hanna (2000) characterized this type of proof construction as the "rote learning of mathematical proofs, devoid of any educational value" (p.10).

Some of the teachers' choices and the resulting classroom microculture may be explained by what Herbst (2002) described as the didactical contract between teacher and student. Herbst claimed that the didactical contract (the teacher's responsibility to teach for understanding and the students' responsibility to learn) will lead the teacher to make choices so that students will be successful at constructing formal proofs. However, although students appear to have success in an immediate context (e.g., in class, on homework), this success is not long lasting. Rather, when students are faced with a variety of problem types for which a logical outline is not provided, they have little independent reasoning ability to draw upon.

The study described here provides a glimpse into the teaching practice of two teachers whose pedagogical strategies are similar. We recognize this as a limitation and acknowledge that further investigations in diverse settings would provide more insight into how other pedagogical approaches may influence the learning of proof. However, the data from this study suggest that teachers need to be cognizant of the power that classroom microculture and pedagogical choices can have in influencing students' understanding of proof. Although there are certainly many factors other than those highlighted that contribute to proof construction ability, this study is a step toward filling the research gap linking pedagogy and student understanding of formal proof.

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